Announcements

1) Valparaiso REU
(sophomores / juniors)

Topics: Stock-option pricing, disease evolution, pattern avoidance in lists

2) Proof of linear independence of eigenvectors associated to distinct eigenvalues worked out in notes.

3) HW #6 vote

Theorem: (diagonalizability)

Let $A \in M_n(\mathbb{C})$. Then A is diagonalizable if and only if $\exists$ a basis for $\mathbb{C}^n$ consisting of eigenvectors of A.

proof: $\Rightarrow$ Suppose

A is diagonalizable,

$A = SDS^{-1}$ for some

diagonal matrix $D \in M_n(\mathbb{C})$

and $S \in M_n(\mathbb{C})$ invertible.

If $d_1, d_2, \ldots, d_n$ are the

diagonal entries of D, then

e_i is an eigenvector of D

for eigenvalue d_i, $1 \leq i \leq n$.

Let $\boxed{v_i = Se_i}$, $1 \leq i \leq n$.

Then

$$
\begin{aligned}
Av_i &= SDS^{-1}(Se_i) \\
&= S(De_i) \\
&= S(d_i e_i) \\
&= d_i Se_i \\
&= d_i v_i
\end{aligned}
$$

This says v_i is an eigenvector of A corresponding to the eigenvalue d_i.

Moreover, $\{v_i\}_{i=1}^n$ is linearly independent since if

$$0 = \sum_{i=1}^n \alpha_i v_i$$

$$= \sum_{i=1}^n \alpha_i S e_i = S\left(\sum_{i=1}^n \alpha_i e_i\right)$$

But S invertible $\Rightarrow$

$\ker(S) = \{0\}$, which

implies $\sum_{i=1}^{n} \alpha_i e_i = 0$.

By linear independence

of $\{e_i\}_{i=1}^{n}$,

$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

This shows $\{v_i\}_{i=1}^{n}$

is linearly independent.

Since $|\{v_i\}_{i=1}^n| = n$,

$\{v_i\}_{i=1}^n$ is a basis

for $\mathbb{C}^n$.

$\Leftarrow$ Suppose A has eigenvectors $\{v_i\}_{i=1}^n$ that are a basis for $\mathbb{C}^n$.

Define $S: \mathbb{C}^n \to \mathbb{C}^n$

$$\boxed{S\left(\sum_{i=1}^n \alpha_i e_i\right) = \sum_{i=1}^n \alpha_i v_i}$$

$\forall \{\alpha_i\}_{i=1}^n \subseteq \mathbb{C}$.

Then S is linear and invertible.

If $Av_i = d_i v_i$, $1 \le i \le n$, let

$$\boxed{D = (a_{i,j})_{i,j=1}^{n} \text{ where } a_{i,j} = \begin{cases} 0, & i \neq j \\ d_i, & i = j \end{cases}}$$

Check that

$$A = SDS^{-1}. \qquad \square$$

Corollary: If $A \in M_n(\mathbb{C})$ has n distinct eigenvalues, then A is diagonalizable.

proof: If $\{\lambda_i\}_{i=1}^{n}$ are the eigenvalues of A, then by a theorem from last class, if v_i is an eigenvector associated to λ_i, $1 \leq i \leq n$,

$\{v_i\}_{i=1}^{n}$ is linearly independent and is therefore a basis for $\mathbb{C}^n$. By previous theorem, A is diagonalizable. $\square$

Naive way to try to Diagonalize

Let $A \in M_n(\mathbb{C})$.

Let $\{\lambda_i\}_{i=1}^m$ be the distinct eigenvalues of A $(m \le n)$.

Let E_i be the eigenspace associated to λ_i, $1 \le i \le m$.

Choose a basis for each E_i.

Then the union of all bases of the E_i's is a basis for $\mathbb{C}^n$!

NO.

The dimension of the eigenspace E_i may be strictly less than the multiplicity of λ_i!

Example 1: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = A.$

Zero is an eigenvalue of A - and it is the only eigenvalue since the characteristic polynomial of A is λ^2.

But the eigenspace associated to $\lambda = 0$ is $\operatorname{span}\{e_1\} \neq \mathbb{C}^2$.

So the dimension of the eigenspace is 1 but the multiplicity of the eigenvalue is 2.

Definition: (semisimple eigenvalue)

Let λ be an eigenvalue for $A \in M_n(\mathbb{C})$, let E be the associated eigenspace. Then λ is semisimple if

$$\boxed{\dim(E) = \text{multiplicity of } \lambda \text{ in the characteristic polynomial of } A}$$

Theorem: (another general theorem)

$A \in M_n(\mathbb{C})$ is diagonalizable if and only if all eigenvalues of A are semisimple.

Proof: $\Rightarrow$ Let A be diagonalizable, $A = SDS^{-1}$. Then as will be shown in HW #6, λ is an eigenvalue of A if and only if λ is an eigenvalue of D.

But if $\lambda_1, \lambda_2, \ldots, \lambda_m$

are the eigenvalues of

D with multiplicities

$k_1, k_2, \ldots, k_m$ $\left(\sum_{i=1}^{m} k_i = n \right)$,

then the characteristic

polynomial for D is

$$\prod_{i=1}^{m} (\lambda - \lambda_i)^{k_i}.$$

Now $\forall\ 1 \leq i \leq m$, the multiplicity of λ_i is k_i, so λ_i occurs on the diagonal of D k_i times $\Rightarrow$

the dimension of the eigenspace associated to λ_i is k_i.

This shows λ_i is semisimple $\forall\ 1 \leq i \leq m$.

$\Leftarrow$ Assume A has eigenvalues $\{\lambda_i\}_{i=1}^m$ with multiplicities $\{k_i\}_{i=1}^m$ summing to n and that λ_i is semisimple $\forall$ $1 \leq i \leq m$. Then if E_i is the associated eigenspace to λ_i, $1 \leq i \leq m$, choose a basis $\mathcal{B}_i$ for each E_i.

By a slight variant of the argument used to show that eigenvectors associated to distinct eigenvalues are linearly independent, we may show that $\mathcal{B}_i$ is linearly independent of $\bigsqcup_{i \neq k} \mathcal{B}_k$, hence $\bigsqcup_{i=1}^{m} \mathcal{B}_i$ is linearly independent.

Since the union is disjoint,

$$\left| \bigsqcup_{i=1}^{m} \mathcal{B}_i \right|$$

$$= \sum_{i=1}^{m} |\mathcal{B}_i| = \sum_{i=1}^{m} k_i = n$$

$$\Rightarrow \bigsqcup_{i=1}^{m} \mathcal{B}_i \text{ is a basis for } \mathbb{C}^n$$

consisting of eigenvectors for A,

which implies A is diagonalizable. $\square$

Special cases

Definition: (adjoint)

Let $A \in M_n(\mathbb{C})$. The adjoint of A is the matrix denoted by $A^* \in M_n(\mathbb{C})$, where

$$\boxed{(A^*)_{i,j} = \overline{(A)_{j,i}}}$$

$\forall \ 1 \leq i, j \leq n$.

Note:

1) A^* is the conjugate transpose of A. So if A is real, $A^* = A^t$.

2) $(AB)^* = B^* A^*$

$\forall\ A, B \in M_n(\mathbb{C})$.

Easy Property of A^*

Let $\langle \cdot, \cdot \rangle$ denote the usual inner product on $\mathbb{C}^n$. Then $\forall A \in M_n(\mathbb{C})$, $x, y \in \mathbb{C}^n$,

$$\boxed{\langle A^* x, y \rangle = \langle x, Ay \rangle}$$

Definition: (normal)

$A \in M_n(\mathbb{C})$ is called

normal if

$$\boxed{A^* A = A A^*}$$